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Invariance principles for semi-stationary sequence of linear processes and applications to ARMA process

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Abstract

Linear semi-stationary processes which are very close to the mixingales considered by McLeish (1975, 1977) are introduced. For these processes an invariance principle is obtained with conditions both simpler and weaker than those retained by McLeish for the mixingales. Furthermore, a particular class of sequences of linear processes called quasi-stationary that gives a framework well-adapted for asymptotic theory of ARMA processes is also considered. For these quasi-stationary sequences, an invariance principle is also obtained and applied to ARMA processes. The results are compared to those obtained by Phillips and Solo (1992) who used the martingale approximating technique introduced by Gordin (1969).

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0. Introduction

Several works in the literature are devoted to extending central limit theorems and invariance principles obtained for martingales to other processes. Gordin (1969) then Scott (1973) (see also Hall and Heyde (1980)) considered strictly stationary and ergodic processes ξ_i and used an approximation technique by martingales to obtain a central limit theorem for the ξ_i's. Gordin's result was extended to a general stationary process by Eagleson (1975). But the condition for the central limit theorem in Gordin (1969) and in Eagleson (1975) is difficult to check (see Hall and Heyde (1980, p. 130)). In Truong-van and Larramendy (1992), we also used a similar martingale approximation technique to obtain asymptotic laws for the mean squares estimators of the autoregressive parameters of unstable ARMA processes.

Phillips and Solo (1992) considered particular linear stationary processes $(\xi_i, i \in \mathbb{Z})$ of the form $\xi_i = \sum_{j=0}^{\infty} \lambda_j \varepsilon_{i-j}$ with the ε_j's being either independent and identically distributed (i.i.d.) real random variables or a martingale difference sequence and where the λ_j's are some sequence of reals. They showed that most asymptotic results on the sample mean and on the sample covariances of the ξ_i's can be proved by using

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a martingale approximation approach entirely similar to that in Gordin (1969) and in Scott (1973). Despite its advantages, this method has shortcomings: First, it only yields asymptotic results on the ξ_i's from the corresponding martingale results but it is not a concept sufficiently general for synthesizing several asymptotic results on the ξ_i's. Thus, in Phillips and Solo (1992), asymptotic results, in particular invariance principles for the sample mean and for the sample covariances of the ξ_i's, require different proofs. This shortcoming can be avoided by taking the approach we propose herein. On the other hand, for sums of the form $\sum_{i=1}^{n} \xi_i$, martingale approximating technique is well suited, but when sums like $\sum_{i=1}^{n} \alpha_{n,i}\, \xi_i$ are to be considered, where the $\alpha_{n,i}$'s are some triangular array of reals, this technique leads to complex calculations (see e.g. Truong-van and Larramendy (1992)) since the remaining term from the martingale approximation can no more cancel simply.

Instead of approximating by martingales, McLeish (1975, 1977) showed another very interesting approach when introducing mixingales and extending Doob's inequality and an invariance principle to mixingales. However, as it will be shown, the mixingales and the conditions in McLeish's invariance principle are inadequate for obtaining asymptotic results for ARMA processes.

Thus, we propose herein a class of linear processes, slightly different from mixingales, called linear semi-stationary processes (LSSP) and show that they provide a well-adapted tool for limit theory of ARMA processes. Following an approach similar to that in McLeish (1975, 1977), we obtain invariance principles for LSSP with simpler and weaker conditions than those considered by McLeish for mixingales.

1. Some preliminaries

Consider a double sequence $(\xi_{n,i}; n \in T, i \in I_n)$ of real random variables (r.v.) in $L_2(\Omega, \mathscr{F}, P)$ equipped with a double sequence $(\mathscr{F}_{n,j}; n \in T, j \in \mathbb{Z})$ of σ-subalgebras in $\mathscr{F}$, where T is a subset of non-negative integers and I_n is a subset in $\mathbb{Z}$. $\mathbb{Z}$ denotes the set of all integers, $(\Omega, \mathscr{F}, P)$ is a probability space and for any $p > 0$, $L_p(\Omega, \mathscr{F}, P)$ is the usual space of r.v. X on $(\Omega, \mathscr{F}, P)$ such that $E|X|^p < \infty$.

Since for any n and i, $\xi_{n,i}$ is in $L_2(\Omega, \mathscr{F}, P)$, its innovations $u_{n,i,j}$ at each time $i + j$ can be defined by $u_{n,i,j} = E_{n,i+j}(\xi_{n,i}) - E_{n,i+j-1}(\xi_{n,i})$, if for each n, $\mathscr{F}_{n,j}$ is increasing with j, where $E_{n,k}$ denotes the conditional expectation relative to $\mathscr{F}_{n,k}$. If the partial sums $\sum_{-m \leq j \leq p} u_{n,i,j}$ converge in some sense to $\xi_{n,i}$ as m and p tend to infinity, we shall say that the $\xi_{n,i}$'s are a sequence of linear processes. In view of the applications to be considered, we restrict ourselves to some sequences of linear processes that are now defined.

Definition 1. The pair $(\xi_{n,i}, \mathscr{F}_{n,j})$ is called a semi-stationary sequence of linear processes if it has the following properties:

(i) For each $n \in T$, $(\mathscr{F}_{n,j}; j \in \mathbb{Z})$ is a filtration, i.e.

$$\mathscr{F}_{n,j} \subset \mathscr{F}_{n,j+1} \subset \mathscr{F}. \tag{1.1}$$

(ii) For each $n \in T$, $i \in I_n$, the series $\sum_{j \in \mathbb{Z}} u_{n,i,j}$ converges in $L_2(\Omega, \mathscr{F}, P)$ to $\xi_{n,i}$, in notation

$$\xi_{n,i} = \sum_{j \in \mathbb{Z}} u_{n,i,j}. \tag{1.2}$$

(iii) For $n \in T$, $i \in T_n$, $j \in \mathbb{Z}$ there are two reals $\alpha_{n,i}$, V_j with $V_j \geqslant 0$ such that

$$E\{(u_{n,i,j})^2\} \leqslant \alpha_{n,i}^2 V_j. \tag{1.3}$$

The notion of semi-stationary sequences of linear processes could be extended to L_p-valued processes for any $p \geqslant 1$ while replacing L_2 by L_p.

In the above definition, we have in fact synthesized two adjacent notions: First, if we take $T = \{0\}$ and $T_0 = \mathbb{Z}$, then the $\xi_{0,i}$'s are reduced to the sequence $(\xi_i = \xi_{0,i}, i \in \mathbb{Z})$ and this will be called a linear semi-stationary process (LSSP). If instead of condition (1.3), the ξ_i's satisfy the following stronger condition: for any integers i and j, there are two reals α_i and V_j with $V_j > 0$ such that $\|u_{i,j}\|_2^2 = \alpha_i^2 V_j$, then it will be called a linear stationary process (LSP), where $u_{i,j}$ denotes $u_{0,i,j}$.

On the other hand, if $T = \{1, 2, \dots\}$ and $I_n = \{1, \dots, k_n\}$ with k_n increasing to infinity with n, then the $\xi_{n,i}$'s are a semi-stationary array of linear processes (SSLP). In most applications, the $\xi_{n,i}$'s are of the form $\xi_{n,i} = \alpha_{n,i}\xi_i$ where the $\alpha_{n,i}$'s are some triangular array of reals with $k_n = n$ and $(\xi_i, i \in \mathbb{Z})$ is a LSP.

So in the following, we shall restrict ourselves to the case of SSLP with $k_n = n$.

Remark 1. (i) Since for any $n \geqslant 1$ and $i \leqslant n$, $(u_{n,i,j}, \mathscr{F}_{n,j};\ j \in \mathbb{Z})$ is a martingale difference sequence (m.d.s.), it follows from (1.2) that $\sum_{j \in \mathbb{Z}} \|u_{n,i,j}\|_2^2 < \infty$ where $\|\cdot\|_2$ denotes the $L_2(\Omega, \mathscr{F}, P)$-norm. From the last relation and the martingale convergence theorem, equality (1.2) holds both in the $L_2(\Omega, \mathscr{F}, P)$-norm and almost surely. In most applications, for each n and i, a m.d.s. $(u_{n,i,j}, \mathscr{F}_{n,j};\ j \in \mathbb{Z})$ is given and the $\xi_{n,i}$'s are defined via relation (1.2). From (1.2), obviously we have for each n, i and j $u_{n,i,j} = E_{n,i+j}(\xi_{n,i}) - E_{n,i+j-1}(\xi_{n,i})$ so that a SSLP is in general given in terms of its innovations via the representation (1.2).

(ii) The concept of semi-stationarity in Definition 1 corresponds in fact to two properties: First, for any n, the filtration associated with each $\xi_{n,i}$ is obtained from the same filtration $(\mathscr{F}_{n,j}, j \in \mathbb{Z})$ by a shift with the step i. The second property is condition (1.3) which means that for any n, i and j, there is a scaling factor $\alpha_{n,i}$ such that the variance of the innovation $u_{n,i,j}$ after renorming by $\alpha_{n,i}^2$ is bounded by a number dependent only on j.

We now discuss some connexions with mixingales. We recall from McLeish (1975, 1977) that a triangular array $(\xi_{n,i}, \mathscr{F}_{n,j};\ n \geqslant 1,\ i \leqslant n,\ j \in \mathbb{Z})$ is a (non-stationary) mixingale if condition (1.1) holds and if there exist two sequences $(\sigma_{n,i})$ and (θ_k) of positive reals such that θ_k converges to 0 as k tends to infinity and for any integers n, i and m with $n \geqslant 1$, $i \leqslant n$

$$\|r_{n,i,m}\|_2 \leqslant \sigma_{n,i}\theta_{|m|} \tag{1.4}$$

where $r_{n,i,m+1} = \xi_{n,i} - E_{n,i+m}(\xi_{n,i})$ if $m \geqslant 0$ and $r_{n,i,m} = E_{n,i+m}(\xi_{n,i})$ if $m \leqslant 0$.

The mixing condition (1.4) implies both relations (1.2) and (1.3). Indeed, condition (1.4) obviously implies (1.2) which in its turn implies that $r_{n,i,m} = \sum_{j \leqslant m} u_{n,i,j}$ if $m \leqslant 0$ and $r_{n,i,m} = \sum_{j \geqslant m} u_{n,i,j}$ if $m \geqslant 1$. Hence for any n, i and j, $\|u_{n,i,j}\|_2 \leqslant \|r_{n,i,j}\|_2$, since $(u_{n,i,j}, \mathscr{F}_{n,j}; j \in \mathbb{Z})$ is a m.d.s. Thus from (1.4), condition (1.3) holds with $\alpha_{n,i}^2 = \sigma_{n,i}^2$ and $V_j = \theta|j|^2$.

Conversely, a SSLP satisfying the additional condition $\sum_{j\in\mathbb{Z}} V_j < \infty$ is a mixingale. It suffices to take $\theta_m^2 = \max(\sum_{j \leqslant -m} V_j, \sum_{j \geqslant m} V_j)$ for any integer $m \geqslant 0$. Then, θ_m decreases to zero as $m \to \infty$ and condition (1.4) holds. These results can be stated as follows.

Proposition 1. *A mixingale is a particular SSLP and conversely, a SSLP such that* $\sum_{j\in\mathbb{Z}} V_j < \infty$ *is a mixingale.*

In order to obtain invariance principles for SSLP or for mixingales, additional conditions are required. McLeish's approach and ours differ from the choice of the additional assumptions, mainly from the choice of the convergence rate of the series of innovations. In McLeish (1975, 1977) the convergence rate is carried on the sequence $(\theta_m, m \geqslant 0)$, hence on the remainders $r_{n,i,m}$. This leads to conditions unapplicable to ARMA processes. We show that it is more judicious to consider additional conditions directly on the innovations and on the convergence rate of the sequence $(V_j, j \in \mathbb{Z})$. Thus, in Theorems 1 and 2, we obtain invariance principles for SSLP with simpler and weaker conditions than those in McLeish (1975, 1977). Furthermore, the conditions in Theorems 1 and 2 can be easily applied to ARMA processes, as it will be illustrated in the following examples.

From now on, we are given a SSLP $\xi_{n,i}$ and for any $n \geqslant 1$ and $t \in [0, 1]$, we define $S_n = \sum_{i=1}^{n} \xi_{n,i}$ and $W_n(t) = S_{[nt]}$ where $[x]$ denotes the integer part of a real x. The weak convergence of the sequence $(W_n, n \geqslant 1)$ towards a Brownian motion W is always meant for the convergence over the space $D = D([0, 1])$ of real functions right-continuous and having left-hand side limits, equipped with the J_1 Skorokhod topology.

Example 1. For any $i \in \mathbb{Z}$, let

$$\xi_i = \sum_{j\in\mathbb{Z}} \lambda_j \varepsilon_{i+j}, \tag{1.5}$$

where $(\lambda_j, j \in \mathbb{Z})$ is a sequence of reals such that $\sum_{j\in\mathbb{Z}} \lambda_j^2 < \infty$ and $(\varepsilon_j, \mathscr{F}_j; j \in \mathbb{Z})$ is a m.d.s. If there is a real $\sigma > 0$ such that for any j

$$E_{j-1}(\varepsilon_j^2) = \sigma^2 \quad \text{a.s.} \tag{1.6}$$

and if $\xi_{n,i} = \alpha_{n,i}\, \xi_i$ where $\{\alpha_{n,i}; n \geqslant 1, i \leqslant n\}$ is a triangular array of reals then the $\xi_{n,i}$'s are obviously a SSLP.

In particular, assume that $\lambda_j = 0$ for $j < 0$ and $\sum |\lambda_j| < \infty$. Let $(Z_i, i \geqslant 1)$ be an unstable ARMA process defined by the equation $U(B)Z_i = \xi_i$ where $U(B)$ is either $(I - B)$ or $(I + B)$ or $(I - 2\cos\theta B + B^2)$ with θ in $(-\pi, \pi)$. If $\alpha_{n,i} = n^{-1/2}\cos(n - i)\omega$ with $\omega \in [-\pi, \pi]$ then invariance principles for the Z_i's are entirely based on the weak convergence of $(W_n, n \geqslant 1)$ (for more details see Chan and Wei (1988) or Truong-van and Larramendy (1992)).

Example 2. Let $(y_i, i \in \mathbb{Z})$ and $(z_i, i \in \mathbb{Z})$ be two regular LSP, that is for any $i \in \mathbb{Z}$

$$y_i = \sum_{j \geqslant 0} \beta_j \varepsilon_{i-j} \quad \text{and} \quad z_i = \sum_{j \geqslant 0} \Psi_j \varepsilon_{i-j} \tag{1.7}$$

with $\sum_{j \geqslant 0} \beta_j^2 < \infty$ and $\sum_{j \geqslant 0} \Psi_j^2 < \infty$ and where $(\varepsilon_j, \mathscr{F}_j; j \in \mathbb{Z})$ is a m.d.s. satisfying condition (1.6).

Define $\xi_{n,i}(\tau) = n^{-1/2} \xi_i(\tau)$ for some integer $\tau \geqslant 0$, with

$$\xi_i(\tau) = y_i z_{i-\tau} - \mu(\tau), \; i \in \mathbb{Z} \quad \text{and} \quad \mu(\tau) = E(y_i z_{i-\tau}) = \sigma^2 \textstyle\sum_{j \geqslant \tau} \beta_j \Psi_{j-\tau}.$$

It can be noted that the $\xi_i(\tau)$'s are a regular process in the sense that for $j \geqslant 1$ the innovations of $\xi_i(\tau)$ at time $i + j$ are null. Thus, for regular processes, we shall take the convention to denote their innovations at time $i - j$ as $u_{i,j}$ instead of $u_{i,-j}$ when $j \geqslant 0$.

If there is some finite positive constant K_2 such that for any $j \geqslant 0$

$$E\{(\varepsilon_j^2 - \sigma^2)^2\} = K_2 \sigma^4 \tag{1.8}$$

then from Lemma 1 below the $\xi_i(\tau)$'s are a LSSP; hence the $\xi_{n,i}(\tau)$'s are a SSLP. Indeed, it suffices to replace Ψ_j by $\Psi_j(\tau)$ and μ by $\mu(\tau)$ in Lemma 1 since the $z_{i-\tau}$'s can be written as $z_{i-\tau} = \sum_{j \geqslant 0} \Psi_j(\tau) \varepsilon_{i-j}$ with $\Psi_j(\tau) = \Psi_{j-\tau}$ if $j \geqslant \tau$ and $\Psi_j(\tau) = 0$ otherwise.

Example 2 has several applications in limit theory of ARMA processes.

For instance, if $\sum_{j \geqslant 0} |\beta_j| < \infty$ and if $(z_i, i \in \mathbb{Z})$ is a regular stationary ARMA (p, ∞) defined by the equation $\varphi(B) z_i = y_i, i \in \mathbb{Z}$, where the zeroes of the polynomial $\varphi(B) = I + \varphi_1 B + \ldots + \varphi_p B^p$ have their modulus greater than one, then the z_i's have the representation (1.7) with $\Psi_j = \sum_{k=0}^{j} v_{j-k} \beta_k$ for $j \geqslant 0$ and $\sum_{j \geqslant 0} |\Psi_j| < \infty$ where the v_k's are the solution of the equation $\varphi(B)\, v_k = 0$ with the initial conditions $v_0 = 1$ and $v_m = 0$ for $m < 0$.

The limit law of the least squares estimators for the parameters φ_i rests upon the weak convergence of the sequences $(W_{n,\tau}, n \geqslant 1)$ where $W_{n,\tau}(t) = \sum_{i=1}^{[nt]} \xi_{n,i}(\tau)$ for $n \geqslant 1$ and $t \in [0,1]$. (For more details see Truong-van and Larramendy (1992)).

Another application of the example is the following. For any i, we take $y_i = z_i$. Then $W_{n,\tau}(t) = n^{1/2} \{R_{[nt]}(\tau) - \mu(\tau)\}$ where the $R_{[nt]}(\tau) = n^{-1} \sum_{i=1}^{[nt]} z_i z_{i-\tau}$ are a version of the sample autocovariance function of the z_i process (see Hall and Heyde (1980, Ch. 6)). Thus, an invariance principle for $R_{[nt]}(\tau)$ is immediately deduced from the convergence of the sequence $(W_{n,\tau}, n \geqslant 1)$.

Lemma 1. (i) *Let $\xi_i = y_i z_i - \mu$, $i \in \mathbb{Z}$ and $\mu = E(y_i z_i) = \sigma^2 \sum_{j \geqslant 0} \beta_j \Psi_j$ where y_i and z_i are defined in* (1.7). *For $j \geqslant 0$, denote the innovations of ξ_i at time $i - j$ by $u_{i,j}$* instead of $u_{i,-j}$. *If conditions* (1.6) *and* (1.8) *hold then $(\xi_i, i \in \mathbb{Z})$ is a LSSP and for $j \geqslant 0$, its innovations at time $i - j$ can be expressed as*

$$u_{i,j} = \beta_j \Psi_j (\varepsilon_{i-j}^2 - \sigma^2) + \varepsilon_{i-j} \{\beta_j E_{i-j-1}(z_i) + \Psi_j E_{i-j-1}(y_i)\} \tag{1.9}$$

or
$$u_{i,j} = \sum_{h \geq 0} \lambda_{j,h}(\varepsilon_{i-j}\varepsilon_{i-j-h} - \sigma^2 \delta_{0,h}) \tag{1.10}$$

with $\|u_{i,j}\|_2^2 \leq \sigma^4 \tilde{V}_j$ *and* $\delta_{k,h}$ *is Kronecker's symbol and*

$$\tilde{V}_j \leq K_2 \lambda_{j,0}^2 + \sum_{h \geq 1} \lambda_{j,h}^2 + 2(K_2 - 1) \sum_{h \geq 1} |\lambda_{j,0}\lambda_{j,h}| \tag{1.11}$$

$\lambda_{j,h} = \beta_j \Psi_j$ *if* $h = 0$ *and* $\lambda_{j,h} = \beta_{j+h} \Psi_j + \beta_j \Psi_{j+h}$ *otherwise.*

(ii) *If in addition for each j,*

$$E_{j-1}(\varepsilon_j^3) = E(\varepsilon_j^3) \text{ a.s.} \tag{1.12}$$

then for any integers i, j, k and l with $j \geq 0$, $l \geq 0$ *and* $h = l - j \geq 0$

$$E(u_{i,j}u_{k,l}) = V_{j,j+h}\, \delta_{i-j,k-l} \tag{1.13}$$

where for any $h \geq 0$

$$V_{j,j+h} = \sigma^4 (K_2 \lambda_{j,0} \lambda_{j+h,0} + \sum_{l \geq 1} \lambda_{j,l} \lambda_{j+h,l}). \tag{1.14}$$

For the convenience of the reader, all the proofs are given in Section 4.

Remark 2. If $(z_i, i \in \mathbb{Z})$ is a regular stationary AR(p) process and each y_i is reduced to ε_i, then the $\xi_i(1)$'s are a m.d.s. So, asymptotic results on AR(p) processes can be derived from the martingale theory. But, if the z_i's are an ARMA(p,q) process with $q \neq 0$, the ξ_i's are no more a m.d.s. Therefore, concepts extending the martingales like LSSP or mixingales are needed for the ARMA case.

2. Invariance principles for SSLP

Theorem 1. *Let* $(\xi_{n,i}, \mathscr{F}_{n,j};\ n \geq 1,\ i \leq n,\ j \in \mathbb{Z})$ *be a SSLP with innovations* $u_{n,i,j}$.

(i) *If the following assumptions hold*

$$\sum_{j \in \mathbb{Z}} V_j^{1/2} < \infty, \tag{H1}$$

$$v_n = \sum_{i=1}^{n} \alpha_{n,i}^2 = O(1), \tag{H2}$$

$$\{(u_{n,i,j}/\alpha_{n,i})^2,\ n \geq 1,\ i \leq n,\ j \in \mathbb{Z}) \text{ uniformly integrable} \tag{H3}$$

then the sequence $(W_n, n \geq 1)$ *is tight in the uniform topology on D.*
(ii) *If in addition for any* $0 \leq \tau < s < t \leq 1$

$$E\left| E_{n,[n\tau]}\left(\sum_{i=[ns]}^{[nt]} \xi_{n,i} \right)^2 - (t-s) \right| \to 0 \quad \text{as } n \to \infty \tag{H4}$$

then $(W_n, n \geq 1)$ *converges weakly over D to a standard Brownian motion as n tends to infinity.*

Remark 3. From both relations (4.4) and (4.9), Theorem 1 is also valid if assumption (H4) is replaced by

$$E\left|E_{n,[n\tau]}\left\{\sum_{i=[ns]}^{[nt]} U_{n,i}([n\tau])\right\}^2 - (t-s)\right| \to 0 \quad \text{as } n \to \infty \tag{2.1}$$

where

$$U_{n,i}([n\tau]) = \xi_{n,i} - E_{n,[n\tau]}(\xi_{n,i}). \tag{2.2}$$

Proposition 2. *For* $j \geqslant 0$ *let* $\theta_j^2 = \max\left\{\sum_{m\leqslant -j} V_m, \sum_{m\geqslant j} V_m\right\}$. *If*

$$\sum_{j=1}^{\infty}\left(\sum_{m=0}^{j}\theta_m^{-2}\right)^{-1/2} < \infty \tag{H1.2}$$

then condition (H1) *holds.*

Corollary 1. (i) *Under assumptions* (H1.2) *and* (H2), *if* $\{(\xi_{n,i}/\alpha_{n,i})^2,\ n \geqslant 1,\ i \leqslant n\}$ *is uniformly integrable then the sequence* $(W_n, n \geqslant 1)$ *is tight in the uniform topology on D.*

(ii) *If in addition condition* (H4) *holds, then* $(W_n, n \geqslant 1)$ *converges weakly over D to a standard Brownian motion as n tends to infinity.*

Remark 4. The corollary is in fact McLeish's invariance principle for non-stationary mixingale (see McLeish (1977, Theorem 2.4)). (McLeish (1977) required the additional condition $\mathrm{Max}\{\sigma_{n,i};\ n \geqslant 1,\ i \leqslant n\} \to 0$ as $n \to \infty$. But, this condition is unnecessary.)

Two other stronger conditions than (H1.2) are considered by McLeish (1975, Theorem 2.5, p. 167; 1977, Remark 2.5) and are respectively

$$\theta_m = \mathrm{O}(n^{1/2}\log n\,(\log\log n)^{1+d})^{-1} \quad \text{with } d > 0 \tag{2.3}$$

and

$$\sum_{m>0}\theta_m^p < \infty \quad \text{with } p < 2. \tag{2.4}$$

However, neither one of these conditions which are basic for the tightness of the sequence $(W_n, n \geqslant 1)$ can be easily expressed in terms of simple and applicable conditions on ARMA processes. Indeed, they depend heavily on the remainders of the series $\sum_{j\in\mathbb{Z}} V_j$ where each term V_j may itself be a sum of a series as shown in Example 2.

Proposition 2 shows that condition (H1) is both much simpler and finer than McLeish's conditions (H1.2), (2.3) and (2.4).

In most applications, the $\xi_{n,i}$'s are of the form $\xi_{n,i} = \alpha_{n,i}\xi_i$ where ξ_i is a LSP with innovations $u_{i,j}$ such that $E(u_{i,j}\,u_{i+h,j-h}) = V_{j,j-h}$. Therefore, we have

$$E(u_{n,i,j}\,u_{n,i+h,j-h}) = \alpha_{n,i}\alpha_{n,i+h}V_{j,j-h}. \tag{2.5}$$

If as in Example 1, $\alpha_{n,i} = n^{-1/2}\cos(n-i)\omega$, $n \geqslant 1$, $i \leqslant n$, $\omega \in (-\pi, \pi)$ then for any integer $h \geqslant 0$,

$$\lim \sum_{i=1}^{n-h} \alpha_{n,i}\alpha_{n,i+h} = (1/2)\cos(h\omega). \tag{2.6}$$

These conditions (2.5) and (2.6) lead to the following class of particular semi-stationary sequences of linear processes.

Definition 2. A triangular array $(\xi_{n,i};\, n \geqslant 1, i \leqslant n)$ of r.v. $\xi_{n,i}$ equipped with a sequence of filtrations $(\mathscr{F}_{n,j};\, n \geqslant 1, j \in \mathbb{Z})$ as in (1.1) is said to be a quasi-stationary sequence of linear processes (QSLP) if it satisfies condition (1.2) and the following properties:

(i) For any integers n, i, j and h with $n \geqslant 1$ and $1 \leqslant i \leqslant n$, there are two reals $\alpha_{n,i,i+h}$ and $V_{j,j-h}$ such that

$$E(u_{n,i,j}\, u_{n,i+h,j-h}) = \alpha_{n,i,i+h} V_{j,j-h}. \tag{QS1}$$

(ii) For any integer $h \geqslant 0$ there is a real $L(h)$ such that

$$\lim_{n\to\infty} \sum_{i=1}^{n-h} \alpha_{n,i,i+h} = L(h). \tag{QS2}$$

(iii) The sequence $(L(h), h \in \mathbb{Z})$ is bounded. (QS3)

Remark 5. (i) A QSLP is a particular SSLP since (QS1) implies (1.3) where the $\alpha_{n,i,i}$'s (resp. the $V_{j,j}$'s) were simply denoted by $\alpha_{n,i}^2$ (resp. by V_j).

(ii) Property (QS2) with $h = 0$ implies condition (H2). Consequently, property (QS3) holds if the $\alpha_{n,i,i+h}$'s satisfy a condition like $\alpha_{n,i,i+h} \leqslant |\alpha_{n,i+h}|\,|\alpha_{n,i}|$ for any n, i and h (as it is the case in (2.5)).

Proposition 3. *Let $(\xi_{n,i}, \mathscr{F}_{n,j};\, n \geqslant 1, i \leqslant n, j \in \mathbb{Z})$ be a QSLP. If condition* (H1) *holds, then for any τ, s and t such that $0 \leqslant \tau \leqslant s \leqslant t \leqslant 1$ we have*

$$\lim_{n\to\infty} E\left(\sum_{i=[ns]}^{[nt]} \xi_{n,i}\right)^2 = \lim_{n\to\infty} E\left\{\sum_{i=[ns]}^{[nt]} U_{n,i}([n\tau])\right\}^2 = \gamma^2(t-s) \tag{2.7}$$

where $U_{n,i}([n\tau])$ is defined in (2.2) *and*

$$\gamma^2 = \sum_{h\in\mathbb{Z}} L(h) \sum_{j\in\mathbb{Z}} V_{j,j-h}. \tag{2.8}$$

From relation (2.7) with $\gamma = 1$ when renorming the $\xi_{n,i}$'s, condition (2.1) amounts to the following:

$$E\left| E_{n,[n\tau]}\left\{\sum_{i=[ns]}^{[nt]} U_{n,i}([n\tau])\right\}^2 - E\left\{\sum_{i=[ns]}^{[nt]} U_{n,i}([n\tau])\right\}^2\right| \to 0 \quad \text{as } n\to\infty. \tag{2.9}$$

According to Remark 3, condition (H4) is equivalent to (2.1). Consequently, expressing (2.9) in terms of the innovations, we obtain the following invariance principle for QSLP.

Theorem 2. Let $(\xi_{n,i}, \mathscr{F}_{n,j}; n \geq 1, i \leq n, j \in \mathbb{Z})$ *be a QSLP. Under Assumptions* (H1) *and* (H3), *if* $\gamma \neq 0$ *and if for* $0 \leq \tau < s < t \leq 1$

$$\lim_{n \to \infty} (p-q) \max_{q \leqslant i \leqslant p} \sum_{j > r-i} \sum_{h \in \mathbb{Z}} E|\Delta_r(i,j,|h|)| = 0 \tag{H4.1}$$

where $\Delta_r(i,j,h) = E_{n,r}(u_{n,i,j}\, u_{n,i+h,j-h}) - E(u_{n,i,j}\, u_{n,i+h,j-h})$ *and where* $r = [n\tau]$, $q = [ns]$ *and* $p = [nt]$ *then* $(W_n, n \geq 1)$ *weakly converges over D to a Brownian motion with variance* γ^2 *defined by relation* (2.8).

3. Applications

We now apply Theorem 2 to the two examples considered before. For the first example, we obtain the following result.

Proposition 4. *Let* $\xi_{n,i} = \alpha_{n,i}\xi_i$ *with* $\alpha_{n,i} = n^{-1/2}\cos(n-i)\omega$, $\omega \in [-\pi, \pi]$, ξ_i *being defined by relation* (1.5) *and the m.d.s.* $\{\varepsilon_j\}$ *satisfying condition* (1.6).

If $\{\varepsilon_j^2, j \in \mathbb{Z}\}$ *is uniformly integrable and if*

$$\sum_{j \in \mathbb{Z}} |\lambda_j| < \infty \tag{3.1}$$

then $(W_n, n \geq 1)$ *weakly converges to a Brownian motion with variance* $\gamma^2 = \sigma^2 \sum_{h \in \mathbb{Z}} L(h) \sum_{j \in \mathbb{Z}} \lambda_j \lambda_{j-h}$ *where* $L(h) = 1$, $(-1)^{|h|}$ *or* $(1/2)\cos(\omega h)$ *according as* $\omega = 0$ *or* $\omega \in \{-\pi, \pi\}$ *or* $\omega \in (-\pi, \pi)$.

The next proposition is an invariance principle for Example 2.

Proposition 5. *Let* $\xi_{n,i} = n^{-1/2}\xi_i$ *for* $n \geq 1$, $i \leq n$ *where the* ξ_i*'s are defined in Lemma* 1 *with* ε_j*'s satisfying conditions* (1.6) *and* (1.12).

If there is some constant K_2 *such that for any integer* j,

$$E_{j-1}\{(\varepsilon_j^2 - \sigma^2)^2\} = K_2\sigma^4 \quad \text{a.s.} \tag{3.2}$$

if $\{\varepsilon_j^4, j \in \mathbb{Z}\}$ *is uniformly integrable and if*

$$\sum_{j \geqslant 0} |\beta_j| < \infty \quad \text{and} \quad \sum_{j \geqslant 0} |\Psi_j| < \infty \tag{3.3}$$

then $(W_n, n \geq 1)$ *converges weakly to a Brownian motion with variance* $\gamma^2 = \sum_{h \in \mathbb{Z}} C_\xi(h)$ *where* C_ξ *is the autocovariance function of the* ξ_i *process,* i.e. *for any* $h \in \mathbb{Z}$, $C_\xi(h) = \sum_{j \geqslant 0} V_{j,j+|h|}$ *with* $V_{j,j+|h|}$ *being defined in Lemma* 1.

Corollary 2. *Let* z_i *be defined by* (1.7) *with the* ε_j*'s satisfying conditions* (1.6), (1.12) *and* (3.2). *For any fixed integer* $\tau \geq 0$, *let* $\xi_{n,i} = n^{-1/2}\ \xi_i(\tau)$ *for* $n \geq 1$, $i \leq n$ *where* $\xi_i(\tau) = z_i z_{i-\tau} - \mu(\tau)$ *and* $\mu(\tau) = \sigma^2 \sum_{j \geqslant \tau} \Psi_{j-\tau}\Psi_j$. *If* $\{\varepsilon_j^4, j \in \mathbb{Z}\}$ *is uniformly integrable and if* $\sum_{j \geqslant 0} |\Psi_j| < \infty$ *then* $(W_n, n \geq 1)$ *converges weakly to a Brownian motion with variance* $\gamma^2 = \sum_{h \in \mathbb{Z}} C_\xi(h)$ *where* C_ξ *is the autocovariance function of the* $\xi_i(\tau)$*'s.*

Remark 6. If in Proposition 4, the ξ_i's are taken to be a regular LSP and if $\omega = 0$ then we have a result quite analogous to invariance principles obtained by Phillips and Solo (1992) for the sample mean of the ξ_i's. For the sample covariance function of the ξ_i's, these authors restrict themselves to the case where the ε_j's are i.i.d. whereas in Corollary 2, a more general case is considered.

4. Proofs

Let us be given a SSLP $(\xi_{n,i}, \mathscr{F}_{n,j};\, n \geqslant 1, i \leqslant n, j \in \mathbb{Z})$ with innovations $u_{n,i,j}$ and for $m \leqslant n$, let $S_{n,m} = \sum_{i=1}^{m} \xi_{n,i}$. First we recall below an inequality due to McLeish (1975, Lemma 6.2).

Lemma 2. *For any $p > 1$ and any sequence $\{c_j\}_{j \geqslant 1}$ of non-negative reals with the convention $c_j^{-1} = 0$ if $c_j = 0$ and $u_{n,i,j} = 0$ for $n \geqslant 1$, $i \leqslant n$, we have*

$$E\left\{\max_{1 \leqslant m \leqslant n} |S_{n,m}|^p\right\} \leqslant \{p/(p-1)\}^p \left(\sum_{j \in \mathbb{Z}} c_j\right)^{p-1} \sum_{j \in \mathbb{Z}} c_j^{1-p} E\{|Z_{n,j}|^p\}$$

where $Z_{n,j} = \sum_{i=1}^{n} u_{n,i,j}$.

Since $E(Z_{n,j}^2) = \sum_{i=1}^{n} \|u_{n,i,j}\|_2^2$, hence from condition (1.3) and Lemma 2 with $p = 2$, we have

$$E\left\{\max_{1 \leqslant m \leqslant n} S_{n,m}^2\right\} \leqslant 4v_n \left(\sum_{j \in \mathbb{Z}} c_j\right)\left(\sum_{j \in \mathbb{Z}} c_j^{-1} V_j\right). \tag{4.0}$$

From this inequality and Lemma 3, the next proposition is obvious.

Proposition 6. *If conditions* (1.3) *and* (H1) *hold then*

$$E\left\{\max_{1 \leqslant m \leqslant n} S_{n,m}^2\right\} \leqslant 4v_n \left(\sum_{j \in \mathbb{Z}} V_j^{1/2}\right)^2$$

where

$$v_n = \sum_{i=1}^{n} \alpha_{n,i}^2. \tag{4.1}$$

Lemma 3. *The two following statements are equivalent.*

(i) *There is a sequence $\{c_j\}_{j \in \mathbb{Z}}$ of non-negative reals with the convention in Lemma 2 such that $\sum_{j \in \mathbb{Z}} c_j < \infty$ and $\sum_{j \in \mathbb{Z}} c_j^{-1} V_j < \infty$.*

(ii) *The series $\sum_{j \in \mathbb{Z}} V_j^{1/2}$ is convergent.*

Proof. The sufficient condition is trivial since it suffices to choose $c_j = V_j^{1/2}$. To prove the necessary condition, let $b_j = c_j^{-1} V_j$. Hence $\sum_{j \in \mathbb{Z}} V_j^{1/2} = \sum_{j \in \mathbb{Z}} b_j^{1/2} c_j^{1/2} \leqslant (\sum_{j \in \mathbb{Z}} b_j)^{1/2} (\sum_{j \in \mathbb{Z}} c_j)^{1/2} < \infty$. □

Proof of Proposition 2. Let $\{c_j\}$ be any non-increasing sequence of positive reals such that $c_j = c_{-j}$. For any $j \geqslant 0$, let $R(-j) = \sum_{m \leqslant -j} V_m$ and $R_+(j) = \sum_{m \geqslant j} V_m$.

Observe that

$$\sum_{j \in \mathbb{Z}} c_j^{-1} V_j = c_0^{-1} R(0) + c_1^{-1} R_+(1) + \sum_{j=1}^{\infty} (c_j^{-1} - c_{j-1}^{-1}) R(-j)$$

$$+ \sum_{j=1}^{\infty} (c_{j+1}^{-1} - c_j^{-1}) R_+(j+1).$$

Therefore

$$\sum_{j \in \mathbb{Z}} c_j^{-1} V_j \leqslant c_0^{-1}(\theta_0^2 + \theta_1^2) + 2 \sum_{j=1}^{\infty} (c_j^{-1} - c_{j-1}^{-1}) \theta_j^2 .$$

According to McLeish (1977, Lemma 3.2) a sequence $\{c_j\}$ can be constructed recursively such that for $j \geqslant 0$, $c_j = (c_j^{-1} - c_{j-1}^{-1})\theta_j^2$ and $\sum_{j \in \mathbb{Z}} c_j \leqslant \sum_{j=0}^{\infty} (\sum_{m=0}^{j} \theta_m^{-2})^{-1/2}$.

Hence from assumption (H1.2), $\sum_{j \in \mathbb{Z}} c_j < \infty$ and $\sum_{j \in \mathbb{Z}} c_j^{-1} V_j < \infty$. So the conclusion follows from Lemma 3. □

Proposition 7. *Under assumptions* (H1), (H2) *and* (H3),

$$\left\{ \max_{1 \leqslant m \leqslant n} S_{n,m}^2 / v_n, n \geqslant 1 \right\}$$

is *uniformly integrable where* v_n *is defined in* (4.1).

Proof. The proof is similar to that in McLeish (1977, Lemma 3.5) but unlike McLeish (1975, 1977), for any positive real c, we decompose $\xi_{n,i}$ into three SSLP as follows:

$$\xi_{n,i} = z_{n,i} + y_{n,i} + x_{n,i}$$

where

$$z_{n,i} = \sum_{|j| > m} u_{n,i,j}, \qquad y_{n,i} = \sum_{|j| \leqslant m} \bar{u}_{n,i,j}, \qquad x_{n,i} = \sum_{|j| \leqslant m} \eta_{n,i,j}$$

with

$$\bar{u}_{n,i,j} = u_{n,i,j} I(|u_{n,i,j}| > c\alpha_{n,i}) - E_{n,i+j-1}[u_{n,i,j} I(|u_{n,i,j}| > c\alpha_{n,i})],$$

$$\eta_{n,i,j} = u_{n,i,j} I(|u_{n,i,j}| \leqslant c\alpha_{n,i}) - E_{n,i+j-1}[u_{n,i,j} I(|u_{n,i,j}| \leqslant c\alpha_{n,i})],$$

where $I(A)$ is the characteristic function of a measurable set A. Let $S_{n,m} = Z_{n,m} + Y_{n,m} + X_{n,m}$ with $Z_{n,m}$ (resp. by $Y_{n,m}$; $X_{n,m}$) be defined as $S_{n,m}$ when replacing $\xi_{n,i}$ by $z_{n,i}$ (resp. by $y_{n,i}$; $x_{n,i}$).

Then for any $\delta > 0$

$$\mathscr{E}_\delta \left\{ \max_{1 \leqslant m \leqslant n} S_{n,m}^2 \right\} \leqslant 9 \left\{ E\left(\max_{1 \leqslant m \leqslant n} Z_{n,m}^2 \right) + E\left(\max_{1 \leqslant m \leqslant n} Y_{n,m}^2 \right) + \mathscr{E}_{\delta/9}\left(\max_{1 \leqslant m \leqslant n} X_{n,m}^2 \right) \right\}$$

where for any random variable $U \geqslant 0$, $\mathscr{E}_\delta(U) = E(U I(U > \delta))$.

From relation (4.0), we have

$$E\left\{\max_{1\leqslant m\leqslant n} Z_{n,m}^2\right\} \leqslant 4v_n\left(\sum_{|j|>m} V_j^{1/2}\right)^2, \tag{4.2}$$

$$E\left\{\max_{1\leqslant m\leqslant n} Y_{n,m}^2\right\} \leqslant 8(2m+1)v_n g(c), \tag{4.3}$$

where $g(c) = \sup\{\mathscr{E}_{c^2}(\bar{u}_{n,i,j}/\alpha_{n,i})^2,\ n\geqslant 1,\ i\leqslant n,\ j\in\mathbb{Z}\}$.

According to both assumptions (H1) and (H3), given a positive constant ε, we may choose an integer $m\geqslant m_0$ and a real $c\geqslant c_0$ such that the right-hand sides in both (4.2) and (4.3) are bounded by ε. The proof ends as in McLeish (1977, Lemma 3.5). □

Remark 7. The same proof is valid for the uniform integrability of

$$\left\{\max_{1\leqslant m\leqslant m+k\leqslant n} (S_{n,m+k}-S_{n,m})^2/v_{n,m,k},\ n\geqslant 1\right\}$$

where $v_{n,m,k} = \sum_{i=m}^{m+k}\alpha_{n,i}^2$. It suffices to replace $S_{n,m}$ by $S_{n,m+k}-S_{n,m}$ and v_n by $v_{n,m,k}$.

Lemma 4. *Let n, p, q and r be any integers with $1\leqslant q\leqslant p\leqslant n$. Let $U_{n,i}(r) = \xi_{n,i} - E_r(\xi_{n,i})$ where $E_{n,r}$ is denoted here simply by E_r. Then we have*

$$E_r\left(\sum_{i=q}^{p}\xi_{n,i}\right)^2 = \left\{\sum_{i=q}^{p}E_r(\xi_{n,i})\right\}^2 + E_r\left\{\sum_{i=q}^{p}U_{n,i}(r)\right\}^2 \text{ a.s.}, \tag{4.4}$$

$$E\left(\sum_{i=q}^{p}\xi_{n,i}\right)^2 = E\left\{\sum_{i=q}^{p}E_r(\xi_{n,i})\right\}^2 + E\left\{\sum_{i=q}^{p}U_{n,i}(r)\right\}^2, \tag{4.5}$$

$$E\left\{\sum_{i=q}^{p}E_r(\xi_{n,i})\right\}^2 \leqslant v_n\sum_{|h|\leqslant p-q}\ \sum_{j\leqslant r-q} V_j^{1/2}V_{j-|h|}^{1/2}, \tag{4.6}$$

$$E\left\{\sum_{i=1}^{q}U_{n,i}(r)\right\}^2 \leqslant v_n\sum_{|h|\leqslant q}\ \sum_{j>r-q} V_j^{1/2}V_{j-|h|}^{1/2}, \tag{4.7}$$

where v_n is defined in (4.1).

Proof. Only relations (4.6) and (4.7) need to be proved since (4.4) and (4.5) are immediate. From the following obvious equalities

$$E_r(\xi_{n,i}) = \sum_{j\leqslant r-i} u_{n,i,j},\quad U_{n,i}(r) = \sum_{j>r-i} u_{n,i,j} \quad\text{and}\quad E(u_{n,i,j}u_{n,k,l}) = 0$$

if $i+j\neq k+l$, it follows that

$$E\left\{\sum_{i=q}^{p}E_r(\xi_{n,i})\right\}^2 = \sum_{|h|\leqslant p-q}\ \sum_{i=q}^{p-|h|}\ \sum_{j\leqslant r-i} E(u_{n,i,j}u_{n,i+|h|,j-|h|}). \tag{4.8}$$

But from assumption (1.3), $E|u_{n,i,j}u_{n,i+h,j-h}| \leqslant |\alpha_{n,i}\alpha_{n,i+h}|V_j^{1/2}V_{j-h}^{1/2}$ hence

$$E\left\{\sum_{i=q}^{p}E_r(\xi_{n,i})\right\}^2 \leqslant \sum_{|h|\leqslant p-q}\left(\sum_{i=q}^{p-|h|}|\alpha_{n,i}\alpha_{n,i+|h|}|\right)\sum_{j\leqslant r-q} V_j^{1/2}V_{j-|h|}^{1/2}.$$

Therefore, relation (4.6) immediately follows. Relation (4.7) is established in a similar way. □

Lemma 5. *Under assumptions* (H1) and (H2) *and with the same notations as in Lemma* 4,

$$\text{if } r \leqslant q \quad \text{then} \left\| \sum_{i=q}^{p} E_r(\xi_{n,i}) \right\|_2 \to 0 \quad \text{as } (q-r) \to \infty, \tag{4.9}$$

$$\text{if } p \leqslant r \quad \text{then} \max\left\{ \left\| \sum_{i=1}^{q} U_{n,i}(r) \right\|_2 ; 1 \leqslant q \leqslant p \leqslant n \right\} \to 0 \quad \text{as } (r-p) \to \infty. \tag{4.10}$$

Proof. From assumption (H1), we have

$$\sum_{h \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} V_j^{1/2} V_{j-h}^{1/2} < \infty. \tag{4.11}$$

Then relation (4.9) follows immediately from (4.6) and (4.11). For part (4.10), observe from (4.7) that we have

$$\max\left\{ \left\| \sum_{i=1}^{q} U_{n,i}(r) \right\|_2^2 ; 1 \leqslant q \leqslant p \leqslant n \right\} \leqslant v_n \sum_{|h| \leqslant p} \sum_{j > r-p} V_j^{1/2} V_{j-|h|}^{1/2}.$$

Therefore, relation (4.10) follows from (4.11). □

Proof of Theorem 1. The idea of the proof is similar to that in McLeish (1977, Theorem 2.4). Under conditions (H1), (H2) and (H3), it follows from Proposition 7 and Remark 7 that the sequence

$$\left\{ \max_{1 \leqslant m < m+k \leqslant n} (S_{n,m+k} - S_{n,m})^2 / v_{n,m,k}, n \geqslant 1 \right\}$$

is uniformly integrable.

Then, from this property and condition (H2), the tightness of $(W_n, n \geqslant 1)$ in the uniform topology in D is deduced from McLeish (1977, Lemma 3.6). Hence, part (i) is proved.

Part (ii) is established by proving that the sequence $(W_n, n \geqslant 1)$ satisfies the conditions (1°a), (2°) and (3°a) in Billingsley (1968, p. 158) that give the asymptotic characterization of a Brownian motion as a particular diffusion process. Conditions (2°) and (3°a) are immediately satisfied (see McLeish, 1975, p. 175) since the sequence

$$\left\{ \max_{1 \leqslant m < m+k \leqslant n} (S_{n,m+k} - S_{n,m})^2 / v_{n,m,k}, n \geqslant 1 \right\}$$

is uniformly integrable.

Only condition (1°a) needs to be shown since its proof differs from McLeish (1975, 1977). Billingsley's condition (1°a) is established if we show like McLeish (1977) that for $m = 1, 2$, for $0 \leqslant t_1 < \cdots < t_k < t_{k+1} \leqslant 1$ and for k arbitrary reals $a_1, \ldots, a_k$

$$E\{\exp(\mathrm{i} Y_n(t_{k-1}) \Delta_{n,m}(t_k))\} \to 0 \quad \text{as } n \to \infty \tag{4.12}$$

where $Y_n(t_{k-1}) = \sum_{j=1}^{r} a_j W_n(t_j)$, $r = [nt_{k-1}]$ and $\Delta_{n,m}(t_k) = \{W_n(t_{k+1}) - W_n(t_k)\}^m - (m-1)(t_{k+1} - t_k)$.

Since for $m = 1,2$ the $\{\Delta_{n,m}(t_k), n \geqslant 1\}$ is uniformly integrable, the main idea of the proof is to show the convergence in probability to zero of the quantities in the brackets in relation (4.12). But

$$E[\Delta_{n,m}(t_k)\exp\{\mathrm{i}Y_n(t_{k-1})\}] = E[\{\exp(\mathrm{i}Y_n(t_{k-1}) - \exp(\mathrm{i}\tilde{Y}_r(t_{k-1})\}\Delta_{n,m}(t_k)] + E[\tilde{\Delta}_{r,m}(t_k)\exp(\mathrm{i}\tilde{Y}_r(t_{k-1})\}]$$

since $E[\Delta_{n,m}(t_k)\exp\{\mathrm{i}\tilde{Y}_r(t_{k-1})\}] = E[\tilde{\Delta}_{r,m}(t_k)\exp(\mathrm{i}\tilde{Y}_r(t_{k-1})]$, where for a r.v. X_n we simply denote $E_{n,r}(X_n)$ by $\tilde{X}_r$.

Then to prove (4.12), it suffices to show for $m = 1,2$ that as $n \to \infty$

$$\| Y_n(t_{k-1}) - \tilde{Y}_r(t_{k-1}) \|_2^2 \to 0, \tag{4.13}$$

$$\| \tilde{\Delta}_{r,m}(t_k) \|_2^2 \to 0. \tag{4.14}$$

Since $Y_n(t_{k-1}) - \tilde{Y}_r(t_{k-1})) = \sum_{j=1}^{d} a_j\{W_n(t_j) - \tilde{W}_r(t_j)\}$ with $d = [nt_{k-2}]$, hence from (4.10) with $p = d$, $\max\{\| W_n(t_j) - \tilde{W}_r(t_j)\|_2, 1 \leqslant j \leqslant k-2\} \to 0$ as $n \to \infty$. Therefore, (4.13) is proved.

On the other hand, from (4.9) with $p = [nt_{k+1}]$ and $q = [nt_k] + 1$, $\| \tilde{\Delta}_{r,1}(t_k)\|_2^2 \to 0$ as $n \to \infty$ and from assumption (H4), $\| \tilde{\Delta}_{r,2}(t_k)\|_2^2 \to 0$ as $n \to \infty$. This completes the proof. □

Proof of Proposition 3. Let $0 \leqslant \tau \leqslant s \leqslant t \leqslant 1$ and let $r = [n\tau]$, $q = [ns]$ and $p = [nt]$. From relation (4.8) and assumption (QS1), we obtain

$$E\left(\sum_{i=q}^{p} \xi_{n,i}\right)^2 = \sum_{|h| \leqslant p-q} \sum_{i=q}^{p-|h|} \sum_{j\in\mathbb{Z}} \alpha_{n,i,i+|h|} V_{i,j-|h|}.$$

Furthermore, for any given positive integer m, proceeding as for (4.6) we obtain

$$\sum_{|h|>m} \sum_{i=q}^{p-|h|} \sum_{j\in\mathbb{Z}} |\alpha_{n,i,i+|h|} V_{j,j-|h|}| \leqslant M \sum_{|h|>m} \sum_{j\in\mathbb{Z}} V_j^{1/2} V_{j-|h|}^{1/2} \tag{4.15}$$

where $M = \sup\{v_n, n \geqslant 1\}$ and v_n is defined in (4.1).

It follows then from (4.11) that for any given $\varepsilon > 0$, we can choose m such that the right-hand side of (4.15) is less than ε.

On the other hand, from (QS2) we have as $n \to \infty$

$$\sum_{|h|\leqslant m} \sum_{i=q}^{p-|h|} \sum_{j\in\mathbb{Z}} \alpha_{n,i,i+|h|} V_{j,j-|h|} \to (t-s) \sum_{|h|\leqslant m} L(h) \sum_{j\in\mathbb{Z}} V_{j,j-|h|}.$$

From both property (QS3) and relation (4.11), we have as $m \to \infty$

$$\sum_{|h|\leqslant m} L(h) \sum_{j\in\mathbb{Z}} V_{j,j-|h|} \to \gamma^2. \tag{4.16}$$

Furthermore from both relations (4.5) and (4.9), as $n \to \infty$ we have

$$\left| E\left(\sum_{i=q}^{p} \xi_{n,i} \right)^2 - E\left\{ \sum_{i=q}^{p} U_{n,i}(r) \right\}^2 \right| \to 0,$$

which completes the proof. □

Proposition 8. *If properties* (QS1), (QS2) *and* (H4.1) *are satisfied, then condition* (H4) *holds.*

Proof. Let $0 \leqslant \tau < s < t \leqslant 1$ and $r = [n\tau]$, $q = [ns]$ and $p = [nt]$. Observe that

$$E_{n,r}\left(\sum_{i=q}^{p} \xi_{n,i} \right)^2 - \gamma^2(t-s) = E_{n,r}\left\{ \sum_{i=q}^{p} U_{n,i}(r) \right\}^2 - E\left\{ \sum_{i=q}^{p} U_{n,i}(r) \right\}^2$$
$$+ E\left\{ \sum_{i=q}^{p} U_{n,i}(r) \right\}^2 - \gamma^2(t-s) + \left\{ \sum_{i=q}^{p} E_{n,r}(\xi_{n,i}) \right\}^2 .$$

Therefore from Lemma 5 and Proposition 3, condition (H4) holds, if

$$E\left| E_{n,r}\left\{ \sum_{i=q}^{p} U_{n,i}(r) \right\}^2 - E\left\{ \sum_{i=q}^{p} U_{n,i}(r) \right\}^2 \right| \to 0 \quad \text{as } n \to \infty . \tag{4.17}$$

It suffices then to show that property (H4.1) implies (4.17). Since $E_{n,r}(u_{n,i,j}u_{n,k,l}) = 0$ a.s if $\min(i+j,\ k+1) > r$ and $i+j \neq k+1$, hence $E_{n,r}\{U_{n,i}(r)U_{n,i+h}(r)\} = \sum_{j>r-i} E_{n,r}(u_{n,i,j}\, u_{n,i+h,j-h})$.

Then

$$E|\Delta(n,p,q,r)| \leqslant \sum_{|h| \leqslant p-q} \sum_{i=q}^{p-|h|} \sum_{j>r-i} E|\Delta_r(i,j,h)|$$
$$\leqslant (p-q) \max_{q \leqslant i \leqslant p} \left\{ \sum_{|h| \leqslant p-q} \sum_{j>r-i} E|\Delta_r(i,j,h)| \right\}$$

where $\Delta(n,p,q,r) = E_{n,r}\{\sum_{i=q}^{p} U_{n,i}(r)\}^2 - E\{\sum_{i=q}^{p} U_{n,i}(r)\}^2$ and $\Delta_r(i,j,h) = E_{n,r}(u_{n,i,j}\, u_{n,i+h,j-h}) - E(u_{n,i,j}\, u_{n,i+h,j-h})$. This completes the proof. □

Theorem 2 is now a consequence of Theorem 1 and Propositions 3 and 8.

Proof of Lemma 1. (i) For any integers i and j with $j \geqslant 0$, we decompose z_i and y_i as follows: $z_i = \zeta_i + E_{i-j}(z_i)$ and $y_i = \eta_i + E_{i-j}(y_i)$. Then $E_{i-j}(z_i y_i) = E_{i-j}(\zeta_i \eta_i) + E_{i-j}(z_i)E_{i-j}(y_i)$ with $E_{i-j}(\zeta_i \eta_i) = \sigma^2 \sum_{l=0}^{j-1} \beta_l \Psi_l$ a.s. if $j \geqslant 1$ and $E_{i-j}(\zeta_i \eta_i) = 0$ otherwise. From the definition $u_{i,j} = E_{i-j}(\xi_i) - E_{i-j-1}(\xi_i)$, hence

$$u_{i,j} = E_{i-j}(z_i y_i) - E_{i-j-1}(z_i y_i)$$
$$= \beta_j(\varepsilon_{i-j} E_{i-j}(z_i) - \sigma^2 \Psi_j) + \Psi_j \varepsilon_{i-j} E_{i-j-1}(y_i).$$

But $E_{i-j}(z_i) = \Psi_j \varepsilon_{i-j} + E_{i-j-1}(z_i)$, hence relation (1.9) holds. From this relation, (1.10) is immediately deduced. A simple calculation leads to relation (1.11).

(ii) Since for any i, $(u_{i,j}, \mathscr{F}_{i-j}; j \geqslant 0)$ is a m.d.s., $E(u_{i,j}\, u_{k,l}) = 0$ if $i - j \neq k - l$. From (1.10), we obtain

$$E(u_{i,j}\, u_{i-h,j+h}) = \lambda_{j,0}\, \lambda_{j+h,0} E\{(\varepsilon_{i-j}^2 - \sigma^2)^2\} + \sigma^4 \sum_{r \geqslant 1} \lambda_{j,r} \lambda_{j+h,r}$$

$$+ \sum_{r \geqslant 1} E\{\varepsilon_{i-j}^3 \varepsilon_{i-j-r})(\lambda_{j,0}\, \lambda_{j+h,r} + \lambda_{j,r} \lambda_{j+h,0}).$$

Then relations (1.13) and (1.14) follow from assumptions (1.8) and (1.12). □

The next lemma is needed for the proof of Propositions 4 and 5.

Lemma 6. *For any $n \geqslant 1$, $i \leqslant n$ let $\xi_{n,i} = \sum_{j \in \mathbb{Z}} \lambda_{i,j}\, u_{n,i,j}$ and $(u_{n,i,j}, \mathscr{F}_{n,i+j}; n \geqslant 1, i \leqslant n, j \in \mathbb{Z})$ be a triangular array of martingale differences relative to a sequence of filtrations $(\mathscr{F}_{n,j}; n \geqslant 1, j \in \mathbb{Z})$ satisfying* (1.1).
Assume that

(1) $\sup \{|\lambda_{i,j}|, i \geqslant 0, j \in \mathbb{Z}\} < \infty$

(2) $\sup_{i \geqslant 0} \limsup_{m \to \infty} \{\sum_{|j| \leqslant m} \lambda_{i,j}^2\, V_j\} = 0.$

If conditions (1.3) *and* (H2) *hold and if $\{(u_{n,i,j}/\alpha_{n,i})^2, n \geqslant 1, i \leqslant n, j \in \mathbb{Z}\}$ is uniformly integrable then so is $\{(\xi_{n,i}/\alpha_{n,i})^2, n \geqslant 1, i \leqslant n\}$.*

Proof. The arguments are similar to those used in Proposition 7. Indeed, let $\xi_{n,i} = z_{n,i} + y_{n,i} + x_{n,i}$ with $z_{n,i} = \sum_{|j| > m} u_{n,i,j}$, $y_{n,i} = \sum_{|j| \leqslant m} \bar{u}_{n,i,j}$ and $x_{n,i} = \sum_{|j| \leqslant m} \tilde{u}_{n,i,j}$, where for a fixed positive constant c,

$$\tilde{u}_{n,i,j} = u_{n,i,j} I(|u_{n,i,j}| > c|\alpha_{n,i}|) - E_{n,i+j-1}\{u_{n,i,j} I(|u_{n,i,j}| > c|\alpha_{n,i}|)\},$$

$$\bar{u}_{n,i,j} = u_{n,i,j} I(|u_{n,i,j}| \leqslant c|\alpha_{n,i}|) - E_{n,i+j-1}\{u_{n,i,j} I(|u_{n,i,j}| \leqslant c|\alpha_{n,i}|)\}.$$

Hence

$$\|z_{n,i}\|_2^2 = \sum_{|j| > m} \lambda_{i,j}^2 \|u_{n,i,j}\|_2^2 \leqslant (\alpha_{n,i})^2 \sum_{|j| > m} (\lambda_{i,j})^2 V_j$$

$$\|y_{n,i}\|_2^2 \leqslant 2(2m+1)(\alpha_{n,i})^2 g(c)$$

and for $\delta > 0$, $\mathscr{E}_{\delta/3}\{(x_{n,i}/\alpha_{n,i})^2\} = 0$ as $\delta/3 > 4(2m+1)^2 c^2$ where $g(c) = \sup\{\mathscr{E}_{c^2} (u_{n,i,j}/\alpha_{n,i})^2, n \geqslant 1, i \leqslant n, j \in \mathbb{Z}\}$ and for any r.v.X, $\mathscr{E}_\delta(X) = E\{XI(X > \delta)\}$.

Hence the proof is achieved as in Proposition 7. □

Proof of Proposition 4. Let $u_{n,i,j}$ be the innovation of $\xi_{n,i}$ at time $i + j$. Since $\xi_{n,i} = \alpha_{n,i}\, \xi_i$ with $\alpha_{n,i} = n^{-1/2} \cos(n - i)\omega$ then $E(u_{n,i,j}\, u_{n,k,l}) = \sigma^2\, \alpha_{n,i} \alpha_{n,i+h} \lambda_j \lambda_{j-h}$ if $k - i = j - l = h$ and $E(u_{n,i,j}\, u_{n,k,l}) = 0$ otherwise.

Then it is immediate that conditions (QS1), (QS2) and (QS3) hold. From condition (3.1), assumption (H1) is obviously satisfied. Then from Proposition 3, we have

$\gamma^2 = \sigma^2 \sum_{h\in\mathbb{Z}} L(h) \sum_{j\in\mathbb{Z}} \lambda_j \lambda_{j-|h|}$ with the $L(h)$'s being defined in Proposition 4. From Lemma 6 and the uniform integrability of $\{\varepsilon_j^2, j \in \mathbb{Z}\}$, it follows that $\{(\lambda_j \varepsilon_{i+j})^2; i,j \in \mathbb{Z}\}$ is also uniformly integrable. So condition (H3) holds since $u_{n,i,j}/\alpha_{n,i} = \lambda_j \varepsilon_{i+j}$.

Furthermore, condition (H4.1) immediately holds since $E_r(u_{n,i,j} u_{n,k,l}) = E(u_{n,i,j} u_{n,k,l})$ a.s. if $\min(i+j, k+l) > r$. Hence from Theorem 2 the proof is complete. □

Proof of Proposition 5. Let the $\lambda_{j,l}$'s be defined as in Lemma 1. It follows from assumption (3.3) on Ψ_j and β_j that

$$\sum_{j\geqslant 0} \sum_{l\geqslant 0} |\lambda_{j,l}| < \infty. \tag{4.18}$$

From both relations (4.18) and (1.13) when taking $h = 0$ and $V_j = V_{j,j}$ in the last relation, assumption (H1) is satisfied.

Since $u_{n,i,j}/\alpha_{n,i} = u_{i,j}$ where from Lemma 1, $u_{i,j} = \sum_{l\geqslant 0} \lambda_{j,l} e_{i,j,l}$ with $e_{i,j,l} = \varepsilon_{i-j}\varepsilon_{i-j-l} - \sigma^2 \delta_{l,0}$ where $\delta_{l,0}$ is Kronecker's symbol, then property (H3) follows from relation (4.18), Lemma 6 and the uniform integrability of $\{\varepsilon_j^4, j \in \mathbb{Z}\}$.

From Lemma 1, the quasi-stationarity conditions are immediately satisfied and $\gamma^2 = \sum_{h\in\mathbb{Z}} C_\xi(h) = \sum_{j\geqslant 0} \sum_{h\in\mathbb{Z}} V_{j,j+|h|}$. So it only remains to verify condition (H4.1). For this, let $0 \leqslant \tau < s < t \leqslant 1$ and $r = [n\tau]$, $q = [ns]$, $p = [nt]$ and decompose the innovations $u_{i,j}$ of the ξ_i process under the form $u_{i,j} = \eta_{i,j}(r) + E_r(u_{i,j})$ where for $i - j > r$, $\eta_{i,j}(r) = \sum_{l<i-j-r} \lambda_{j,l} e_{i,j,l}$ is the r-truncation of $u_{i,j}$ with the corresponding remainder $E_r(u_{i,j}) = \varepsilon_{i-j} R_{i,j}(r)$ where $R_{i,j}(r) = \beta_j E_r(z_i) + \Psi_j E_r(y_i) = \sum_{l\geqslant i-j-r} \lambda_{j,l} \varepsilon_{i-j-l}$. Let $\Delta_r(i,j,h) = E_r(u_{i,j} u_{i+h,j-h}) - E(u_{i,j} u_{i+h,j-h})$. Then $\Delta_r(i,j,h) = \sum_{k=1}^{3} \delta_k(r,i,j,h)$ where for $k = 1,2$ and 3, $\delta_k(r,i,j,h) = E_r\{Z_k(r,i,j,h)\} - E\{Z_k(r,i,j,h)\}$ with

$$Z_1(r,i,j,h) = \eta_{i,j}(r) \eta_{i+h,j+h}(r),$$

$$Z_2(r,i,j,h) = \sigma^2 R_{i,j}(r) R_{i+h,j+h}(r),$$

$$Z_3(r,i,j,h) = R_{i,j}(r) E_r\{\varepsilon_{i-j} \eta_{i+h,j+h}(r)\} + R_{i+h,j+h}(r) E_r\{\varepsilon_{i-j} \eta_{i,j}(r)\}.$$

Let $\Delta(r,q,p) = \sum_{|h|\leqslant p-q} \max_{q\leqslant i\leqslant p} \sum_{j<r-i} E|\Delta_r(i,j,|h|)|$ then $\Delta(r,q,p) \leqslant \sum_{k=1}^{3} \Delta_k(r,q,p)$ where for $k = 1,2,3$ $\Delta_k(r,q,p) = \sum_{|h|\leqslant p-q} \max_{q\leqslant i\leqslant p} \sum_{j<r-i} E|\delta_k(r,i,j,|h|)|$.

Since $u_{n,i,j} = n^{-1/2} u_{i,j}$, condition (H4.1) is established if we show that for $k = 1,2,3$ $\Delta_k(r,q,p) \to 0$ as $n \to \infty$.

From assumption (3.2), $\delta_1(r,i,j,|h|) = 0$ a.s., so $\Delta_1(r,q,p) = 0$ a.s. A simple calculation shows that

$$\delta_2(r,i,j,h) = \sigma^2 \left\{ \sum_{l\geqslant i-j-r} \sum_{m\geqslant i-j-r} \lambda_{j,l} \lambda_{j+h,m} \varepsilon_{i-j-l} \varepsilon_{i-j-m} - \sigma^2 \sum_{l\geqslant i-j-r} \lambda_{j,l} \lambda_{j+h,l} \right\}$$

hence

$$\Delta_2(r,q,p) \leqslant 2\sigma^4 \left(\sum_{j=0}^{p-r} \sum_{l\geqslant q-r} |\lambda_{j,l-j}| \right) \left(\sum_{|h|\leqslant p-q} \sum_{m\geqslant q-r} |\lambda_{j+|h|,m-j+|h|}| \right).$$

Therefore it follows from (3.3) that as $n \to \infty$ $\Delta_2(r,q,p) \to 0$. It can be assumed without loss of generality that $Z_3(r,i,j,h) = R_{i+h,j+h}(r) E_r\{\varepsilon_{i-j}\eta_{i,j}(r)\}$. From Hölder's inequality and assumptions (1.6) and (3.2), we have for any integers i and j with $i > r$ $E|\varepsilon_i E_r(\varepsilon_j^3)| \leqslant \|\varepsilon_i\|_4 \, \|\varepsilon_j^3\|_{4/3} \leqslant \sup \ \{E(\varepsilon_j^4);\ j \in \mathbb{Z}\} = (K_2 - 1)\sigma^4$ where for $m > 0$, $\|\varepsilon_j\|_m^m = E(|\varepsilon_j|^m)$. Then it is immediate that $\Delta_3(r,q,p) \leqslant (K_2 - 1)\sigma^4$ $(\sum_{j<q-r}\sum_{l \leqslant p-r} |\lambda_{j,l-j}|)(\sum_{|h| \leqslant p-q}\sum_{m \geqslant q-r} |\lambda_{j+|h|,m-j+|h|}|)$. Hence from (3.3), $\Delta_3(r,q,p) \to 0$ as $n \to \infty$. So the proof is achieved. $\square$

Corollary 2 is deduced from Proposition 5 when replacing β_j by $\Psi_j(\tau)$ with $\Psi_j(\tau) = \Psi_{j-\tau}$ if $j \geqslant \tau$ and $\Psi_j(\tau) = 0$ otherwise.

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